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Hadamard product of analytic functions and some special regions and curves

Krzysztof Piejko and Janusz Sokół*

Dedicated to Professor Hari M Srivastava

*Correspondence: jsokol@prz.edu.pl
Department of Mathematics,
Rzeszów University of Technology,
Al. Powstańców Warszawy 12,
Rzeszów, 35-959, Poland**Abstract**

In this paper we present some new applications of convolution and subordination in geometric function theory. The paper deals with several ideas and techniques used in this topic. Besides being an application to those results, it provides interesting corollaries concerning special functions, regions and curves.

MSC: Primary 30C45; secondary 30C80**Keywords:** Hadamard product; univalent functions; convex functions; closed convex hull; convolution; subordination

1 Introduction

By \mathcal{H} we denote a class of analytic functions in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Let \mathcal{K} denote the subclass of \mathcal{H} consisting of functions normalized by $f(0) = 0$, $f'(0) = 1$ and satisfying

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad \text{for } z \in \Delta. \quad (1.1)$$

Of course the functions from \mathcal{K} map the unit disc onto convex domains.

We say that the function $f \in \mathcal{H}$ is subordinate to the function $g \in \mathcal{H}$ in the unit disc Δ (written $f \prec g$) if and only if there exists an analytic function $w \in \mathcal{H}$ such that $w(0) = 0$, $|w(z)| < 1$ and $f(z) = g(w(z))$ for $z \in \Delta$. Therefore $f \prec g$ in Δ implies $f(\Delta) \subset g(\Delta)$. If, additionally, g is univalent in Δ , then

$$f \prec g \quad \text{if and only if} \quad f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta). \quad (1.2)$$

Recall that the Hadamard product or convolution of two power series,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

is defined as

$$(f * g)(z) = f(z) * g(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

The convolution has the algebraic properties of ordinary multiplication. We now look at some problems on convolution and at some of the relations between the convolution and the subordination. One can consider the following problems.

Problem 1 Find the subclasses \mathcal{A} , \mathcal{B} , \mathcal{C} of the class \mathcal{H} such that

$$\mathcal{A} * \mathcal{B} = \mathcal{C},$$

where

$$\mathcal{A} * \mathcal{B} = \{h(z) = (f * g)(z) \text{ for all } f \in \mathcal{A} \text{ and for all } g \in \mathcal{B}\}.$$

Problem 2 Let the sets $M \subset \mathbb{C}$ and $N \subset \mathbb{C}$ be given. Find a set $K \subset \mathbb{C}$, as small as possible, such that

$$f(\Delta) \subset M \quad \text{and} \quad g(\Delta) \subset N \quad \Rightarrow \quad (f * g)(\Delta) \subset K \quad \text{for all } f, g \in \mathcal{D},$$

where \mathcal{D} is some subclass of \mathcal{H} .

If $M = F(\Delta)$ and $N = G(\Delta)$, where $F, G \in \mathcal{H}$ are univalent in Δ , then, by (1.2), Problem 2 above becomes the following one:

Problem 3 Let the univalent functions F, G be given. Find a function H , $H(\Delta) = K$, and a class $\mathcal{D} \subset \mathcal{H}$ such that

$$f \prec F \quad \text{and} \quad g \prec G \quad \Rightarrow \quad f * g \prec H \quad \text{for all } f, g \in \mathcal{D}.$$

Ruscheweyh and Sheil-Small [1] proved the Pólya-Schoenberg conjecture that the class of convex univalent functions is preserved under convolution, namely $\mathcal{K} * \mathcal{K} = \mathcal{K}$. They proved also that the class of starlike functions and the class of close-to-convex functions are closed under convolution with the class \mathcal{K} . Another solution of Problem 1 is the following theorem.

Theorem A (see [2]) *Let us denote*

$$\mathcal{P}_x = \{p \in \mathcal{H} : p(0) = 1 \text{ and } \Re\{p(z)\} > x\}.$$

Suppose that $\alpha \leq 1$, $\beta \leq 1$. Then

$$\mathcal{P}_\alpha * \mathcal{P}_\beta = \mathcal{P}_\gamma,$$

where $\gamma = 1 - 2(1 - \alpha)(1 - \beta)$.

Problem 2, when the sets M and K are discs $\mathcal{R}(s_i, R_i)$ with center s_i and radius R_i , was considered in [3] and in other papers. One of results obtained there is the following theorem.

Theorem B (see [3]) *Let $s_i \in \mathbb{C}$, $R_i \in \mathbb{R}$; $i = 1, 2$. If $f, g \in \mathcal{H}$ and $f(0) = a$, $g(0) = b$, then the following implication holds:*

$$f(\Delta) \subset \mathfrak{K}(s_1, R_1) \quad \text{and} \quad g(\Delta) \subset \mathfrak{K}(s_2, R_2) \quad \Rightarrow \quad (f * g)(\Delta) \subset \mathfrak{K}(s, R) \quad \text{for all } f, g \in \mathcal{H},$$

where

$$s = ab + m(s_1 - a)(s_2 - b), \quad R = mR_1R_2, \quad m = \frac{(R_1^2 - |s_1 - a|^2)(R_2^2 - |s_2 - a|^2)}{R_1^2R_2^2 - |s_1 - a|^2|s_2 - a|^2}.$$

In this paper we shall look for a solution of the following modified Problem 3.

Problem 4 Find a function $F \in \mathcal{H}$ and a class $\mathcal{D} \subset \mathcal{H}$ such that

$$g \prec f \quad \Rightarrow \quad g * F \prec f * F \quad \text{for all } f \in \mathcal{D}, g \in \mathcal{H}.$$

The next theorem is a solution of Problem 4.

Theorem C (see [4]) *Assume that $f \in \mathcal{H}$, $f'(0) \neq 0$, and that it satisfies*

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > -\frac{1}{2} \quad \text{for } z \in \Delta.$$

If $g \prec f$, then

$$L[g] \prec L[f],$$

where

$$L[f](z) = \frac{2}{z} \int_0^z f(t) dt$$

is the Libera operator [5].

If $f(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + \dots$, then we have

$$\begin{aligned} L[f](z) &= \frac{2}{z} \int_0^z f(t) dt = \sum_{n=0}^{\infty} \frac{2a_n}{n+1} z^n \\ &= f(z) * \left(\frac{2}{z} \log \frac{1}{1-z} \right). \end{aligned}$$

Therefore, Theorem C may be written as

$$\begin{aligned} g \prec f \quad \Rightarrow \quad g * \left(\frac{2}{z} \log \frac{1}{1-z} \right) &\prec f * \left(\frac{2}{z} \log \frac{1}{1-z} \right) \\ \text{for all } f \in \mathcal{K}(-1/2), g \in \mathcal{H}, \end{aligned} \tag{1.3}$$

where

$$\mathcal{K}(\alpha) := \left\{ f \in \mathcal{H} : f'(0) \neq 0 \text{ and } \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \text{ for } z \in \Delta \right\}. \tag{1.4}$$

Theorem C improves an earlier result from [6] with the stronger assumption $f \in \mathcal{K}(0)$ and the same conclusion. Moreover, if we assume more, that $f \in \mathcal{K}(0)$ and f is univalent in Δ , then in (1.3) instead of $F(z) = -2/z \log(1-z)$ we can put any convex univalent function. This is contained in the following result due to Ruscheweyh and Stankiewicz [7].

Theorem D (see [7]) *Let F and G be convex univalent in Δ . Then, for all functions $f, g \in \mathcal{H}$,*

$$f \prec F \quad \text{and} \quad g \prec G \quad \implies \quad f * g \prec F * G. \quad (1.5)$$

This relationship allows us to obtain several subordination results about convolution. Note that there are no assumptions about the normalization of functions. It is one of the solutions of Problem 3. Many of the convolution properties were studied by Ruscheweyh in [8], and they have found many applications in various fields. The book [8] is also an excellent survey of the results. For the recent results on the Hadamard product in geometric function theory, see [9–13].

2 A family of functions

Throughout this section we consider the family of analytic functions

$$F_s(z) = \frac{z}{1-sz^2} \quad (z \in \Delta), \quad (2.1)$$

where $s \in [0, 1]$. In Section 3 we prove that for given $s \in [0, 1]$ the curve $F_s(e^{i\varphi})$, $\varphi \in [0, 2\pi]$ is the Booth lemniscate. Let \mathcal{S}^* denote a class of starlike functions consisting of functions f such that $zf' \in \mathcal{K}$.

Lemma 2.1 *The function F_s given by (2.1) is a starlike univalent function for $s \in [0, 1]$.*

Proof We have

$$\Re \left\{ \frac{zF'_s(z)}{F_s(z)} \right\} = \Re \left\{ \frac{1+sz^2}{1-sz^2} \right\} > 0 \quad \text{for all } z \in \Delta.$$

Thus, it is obvious that F_s is a starlike univalent function for $s \in [0, 1]$. □

Lemma 2.2 *Let the function F_s be of the form (2.1), and let $\mathcal{K}(-1/2)$ be given by (1.4). If $|s| \leq 6 - \sqrt{33} \approx 0.255437$, then $F_s \in \mathcal{K}(-1/2)$.*

Proof Assume that $sz^2 = r(\cos t + i \sin t)$, $0 \leq r < 1$, $0 \leq t \leq 2\pi$ and set

$$H(t, r) := \frac{zF''_s(z)}{F'_s(z)} + 2 = \frac{6(1+sz^2) - 4}{1-(sz^2)^2} = \frac{2 + 6r \cos t + 6ir \sin t}{1 - r^2 \cos 2t - ir^2 \sin 2t}. \quad (2.2)$$

By (1.4) if $\Re\{H(t, r)\} > 1/2$ for all $r \in [0, r_0]$, $t \in [0, 2\pi]$, then $F_s \in \mathcal{K}(-1/2)$ for all $|s| \leq r_0$. After some calculations, we obtain

$$\Re\{H(t, r)\} = \frac{-4r^2x^2 + 6r(1-r^2)x + 2(1+r^2)}{(1+r^2)^2 - 4r^2x^2} := \frac{n(x, r)}{d(x, r)}, \quad (2.3)$$

where $x = \cos t$. Thus we shall find r_0 in $[0, 1]$ such that

$$\frac{n(x, r)}{d(x, r)} > 1/2 \quad \text{for all } r \in [0, r_0) \text{ and for all } x \in [-1, 1]. \quad (2.4)$$

It is easy to verify that $d(x, r) > 0$ for all $r \in [0, 1]$ and for all $x \in [-1, 1]$, and that $n(-1, r) = 2(1 - r^2)(1 - 3r) \leq 0$ for all $r \in [1/3, 1]$. Hence, to find r_0 in $[0, 1]$ satisfying (2.4), we may restrict our considerations to the interval $[0, 1/3]$. Then, analyzing $n(x, r)$ as a function of one variable x , we get

$$\min\{n(x, r) : -1 \leq x \leq 1\} = n(-1, r) = 2(1 - r^2)(1 - 3r) > 0$$

for all $r \in [0, 1/3]$. Moreover, we see that

$$\min\{d(x, r) : -1 \leq x \leq 1\} = d(-1, r) = (1 - r^2)^2 > 0.$$

Therefore, if $0 \leq r < 1/3$, then

$$\min\left\{\frac{n(x, r)}{d(x, r)} : x \in [-1, 1]\right\} = \frac{2(1 - 3r)}{1 - r^2}.$$

To find r_0 in $[0, 1/3]$ satisfying (2.4), it is sufficient to solve the inequality

$$\frac{2(1 - 3r)}{1 - r^2} > \frac{1}{2},$$

which is true for $0 \leq r \leq r_0 = 6 - \sqrt{33}$. □

Moreover, analogous consideration as in the above proof leads us to the fact that $F_s \notin \mathcal{K}(-1/2)$ for $s > 6 - \sqrt{33} \approx 0.255437$. In Lemma 3.2 we show that $F_s \in \mathcal{K}$ for $0 \leq s \leq 3 - 2\sqrt{2} \approx 0.1715$.

Theorem 2.3 *If $0 \leq s \leq 6 - \sqrt{33} \approx 0.255437$ and $f \in \mathcal{H}$, then*

$$f \prec F_s \implies L[f] \prec L[F_s], \quad (2.5)$$

where

$$L[f](z) = \frac{2}{z} \int_0^z f(t) dt$$

is the Libera operator.

Proof By Lemma 2.2 we obtain

$$\Re\left\{1 + \frac{zF_s''(z)}{F_s'(z)}\right\} > -\frac{1}{2} \quad \text{for } z \in \Delta.$$

Therefore, using Theorem C, we get (2.5). □

Corollary 2.4 *If $0 \leq s \leq 6 - \sqrt{33}$ and*

$$f(z) = \sum_{n=1}^{\infty} a_n z^n,$$

then

$$\sum_{n=1}^{\infty} a_n z^n < \sum_{n=1}^{\infty} s^{n-1} z^{2n-1} \implies \sum_{n=1}^{\infty} \frac{a_n}{n+1} z^n < \sum_{n=1}^{\infty} \frac{s^{n-1}}{n+1} z^{2n-1}.$$

It is known [14] that if

$$\sum_{n=0}^{\infty} \alpha_n z^n < \sum_{n=0}^{\infty} \beta_n z^n,$$

then

$$\sum_{n=0}^k |\alpha_n|^2 \leq \sum_{n=0}^k |\beta_n|^2.$$

Hence, for $0 \leq s \leq 6 - \sqrt{33}$, Corollary 2.4 gives also that

$$\begin{aligned} \sum_{n=1}^{\infty} a_n z^n < F_s(z) &\implies \sum_{n=1}^{\infty} \frac{|a_n|^2}{(n+1)^2} \leq \sum_{n=1}^{\infty} \frac{s^{2(n-1)}}{(n+1)^2} = \frac{-1}{s^2} + \frac{1}{s^2} \int_0^{\infty} \frac{t}{e^t - s^2} dt \\ &= \frac{\Phi(s^2, 2, 1) - 1}{s^2}, \end{aligned}$$

because for $\Re\{a\} > 0$, $-1 \leq x < 1$, $\Re\{c\} > 0$,

$$\Phi(x, c, a) = \sum_{k=0}^{\infty} \frac{x^k}{(k+a)^c} = \frac{1}{\Gamma(c)} \int_0^{\infty} \frac{t^{c-1} e^{-at}}{1 - xe^{-t}} dt.$$

Theorem 2.5 *If $0 \leq s_i \leq 6 - \sqrt{33}$, $i = 1, 2$, and the functions g_1 and g_2 are in the class \mathcal{H} , then*

$$g_1 < F_{s_1} \quad \text{and} \quad g_2 < F_{s_2} \implies L[g_1] * L[g_2] < L[L[F_{s_1 s_2}]]. \quad (2.6)$$

Proof By Theorem 2.3 we have $L[g_1] < L[F_{s_1}]$ and $L[g_2] < L[F_{s_2}]$. By Lemma 2.1 $F_s \in \mathcal{S}^*$, $s \in [0, 1]$, so the functions $L[F_{s_1}]$, $L[F_{s_2}]$ are in \mathcal{K} because $L[\mathcal{S}^*] = \mathcal{K}$. Using Theorem D, we obtain

$$L[g_1] * L[g_2] < L[F_{s_1}] * L[F_{s_2}]$$

but $L[F_{s_1}] * L[F_{s_2}] = L[L[F_{s_1 s_2}]]$ because

$$F_{s_1}(z) * F_{s_2}(z) = \frac{z}{1 - s_1 z^2} * \frac{z}{1 - s_2 z^2} = \frac{z}{1 - s_1 s_2 z^2} = F_{s_1 s_2}(z). \quad \square$$

3 A family of sets

In this section we investigate the properties of a one-parameter family of the sets $D(s)$, $s \in [0, 1]$, such that

$$D(s) = \left\{ x + iy \in \mathbb{C} : (x^2 + y^2)^2 - \frac{x^2}{(1-s)^2} - \frac{y^2}{(1+s)^2} < 0 \right\}, \quad (3.1)$$

when $s \in [0, 1)$ and

$$D(1) = \{ x + iy \in \mathbb{C} : x + iy \neq it \text{ for all } t \in (-\infty, -1) \cup (1, \infty) \}. \quad (3.2)$$

Lemma 3.1 *Let $s \in [0, 1]$, then $D(s) = F_s(\Delta)$, where*

$$F_s(z) = \frac{z}{1 - sz^2} \quad (z \in \Delta).$$

Proof The function F_s is analytic in the unit disc, so for the proof we need to find an image of the circle $|z| = 1$ under the function F_s . For $s \in [0, 1)$ and for $\varphi \in [0, 2\pi)$, we have

$$\begin{aligned} F_s(e^{i\varphi}) &= \frac{e^{i\varphi}}{1 - se^{2i\varphi}} \cdot \frac{1 - se^{-2i\varphi}}{1 - se^{-2i\varphi}} = \frac{e^{i\varphi} - se^{-i\varphi}}{1 + s^2 - s(e^{2i\varphi} + e^{-2i\varphi})} \\ &= \frac{(1-s)\cos\varphi + i(1+s)\sin\varphi}{1 + s^2 - 2s\cos 2\varphi}. \end{aligned} \quad (3.3)$$

Let us define

$$x = \Re\{F_s(e^{i\varphi})\} = \frac{(1-s)\cos\varphi}{1 + s^2 - 2s\cos 2\varphi} \quad (3.4)$$

and

$$y = \Im\{F_s(e^{i\varphi})\} = \frac{(1+s)\sin\varphi}{1 + s^2 - 2s\cos 2\varphi}. \quad (3.5)$$

Then, after some calculations, we can obtain from (3.4) and (3.5) that

$$x^2 + y^2 = \frac{1}{1 + s^2 - 2s\cos 2\varphi} \quad \text{and} \quad (x, y) \neq (0, 0). \quad (3.6)$$

Therefore, using the relations (3.3)-(3.5), we can find that $F_s(e^{i\varphi})$ is an algebraic curve of order four whose equation in orthogonal Cartesian coordinates is

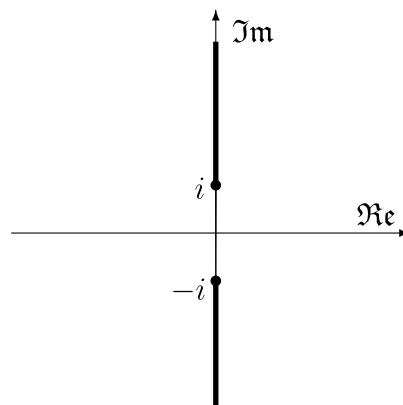
$$(x^2 + y^2)^2 - \frac{x^2}{(1-s)^2} - \frac{y^2}{(1+s)^2} = 0 \quad \text{and} \quad (x, y) \neq (0, 0). \quad (3.7)$$

Thus, because $F_s(0) = 0$ for $s \in [0, 1)$, the set $F_s(\Delta)$ is bounded by the curve (3.7). Moreover, for $s = 1$, the function F_s becomes

$$F_1(z) = \frac{z}{1 - z^2}$$

and it is easy to see that $F_1(\Delta) = D(1)$, see Figure 1. Then the proof is completed. \square

Figure 1 $F_1(\Delta) = D(1)$.



Notice that a curve described by

$$(x^2 + y^2)^2 - (n^4 + 2m^2)x^2 - (n^4 - 2m^2)y^2 = 0 \quad \text{and} \quad (x, y) \neq (0, 0) \quad (3.8)$$

is called the Booth lemniscate, named after Booth [15, 16]. The Booth lemniscate is called elliptic if $n^4 > 2m^2$, while for $n^4 < 2m^2$ it is termed hyperbolic. Thus it is clear that the curve (3.7) is the Booth lemniscate of elliptic type. The Booth lemniscate is a special case of a Persian curve. The Booth lemniscate of elliptic type can be described geometrically in the following equivalent ways.

1. Suppose that $s \in [0, 1)$. Let $\mathcal{C}(S, R)$ be a circle with the center S and the length of the radius R such that

$$S = \left(\frac{\sqrt{s}}{1-s^2}, 0 \right) \quad \text{and} \quad R = \frac{1}{2(1-s)}.$$

A ray is drawn from the point x on the circle $\mathcal{C}(S, R)$ through the origin o that cuts also the circle $\mathcal{C}(S, R)$ at the point z . A point v is also on the ray and satisfies

$$|ov| = |xz|. \quad (3.9)$$

If the point x goes along the circle $\mathcal{C}(S, R)$, then the point v describes the curve (3.7), see Figure 2.

2. Let $s \in [0, 1)$. Then the curve (3.7) consists of points M such that

$$|F_1 M|^2 |F_2 M|^2 = \frac{1+s^2}{(1-s^2)^2} |OM|^2 + \frac{s^2}{(1-s^2)^4},$$

where $O = (0, 0)$ and the points F_1, F_2 are the focuses

$$F_1 \left(\frac{-\sqrt{s}}{1-s^2}, 0 \right), \quad F_2 \left(\frac{\sqrt{s}}{1-s^2}, 0 \right),$$

see Figure 3.

We say that a closed curve γ is convex when it is boundary of a convex bounded domain. Otherwise, we say that the curve γ is concave.

Figure 2 $D(s)$, $s \in [0, 3 - 2\sqrt{2}]$.

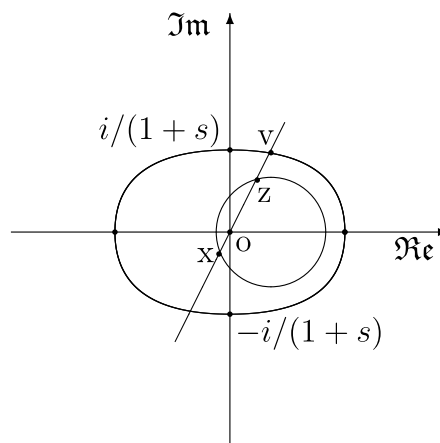
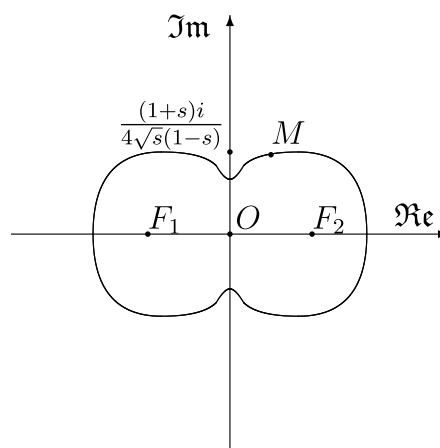


Figure 3 $D(s)$, $s \in (3 - 2\sqrt{2}, 1)$.



Lemma 3.2 Suppose that F_s is given by (2.1). If $0 \leq s \leq 3 - 2\sqrt{2} \approx 0.1715$, then the curve $F_s(e^{i\varphi})$, $\varphi \in [0, 2\pi)$ is convex and

$$\max \{ \Im \{ F_s(e^{i\varphi}) \} : \varphi \in [0, 2\pi) \} \quad (3.10)$$

is attained at one point only. If $s \in (3 - 2\sqrt{2}, 1)$, then the curve $F_s(e^{i\varphi})$, $\varphi \in [0, 2\pi)$ is concave and (3.10) is attained twice. Moreover, in both cases this curve is symmetric with respect to both axes.

Proof If $s = 0$, then the curve $F_s(e^{i\varphi})$, $\varphi \in [0, 2\pi)$ becomes a circle and it is clear that (3.10) is attained one time only. Suppose, in the sequel, that $s \in (0, 1)$. From (2.2) and (2.3) we have

$$\Re \left\{ \frac{e^{i\varphi} F_s''(e^{i\varphi})}{F_s'(e^{i\varphi})} + 1 \right\} = \frac{1 - s^4 + 6s(1 - s^2) \cos \varphi}{(1 + s^2)^2 - 4s^2 \cos^2 \varphi}.$$

The denominator is positive, hence for the convexity of $F_s(e^{i\varphi})$ we need the nominator to be positive. Since $s^2 - 6s + 1 > 0$ for $0 \leq s \leq 3 - 2\sqrt{2}$, then the nominator is positive because

$$\begin{aligned} 1 - s^4 + 6s(1 - s^2) \cos \varphi &\geq 1 - s^4 - 6s(1 - s^2) \\ &= (1 - s^2)(s^2 - 6s + 1). \end{aligned}$$

From (3.5) we have

$$\Im\{F_s(e^{i\varphi})\} = \frac{(1+s)\sin\varphi}{1+s^2-2s\cos 2\varphi} := \frac{(1+s)x}{1+s^2-2s(1-2x^2)} := f(x).$$

Then $f'(x) = 0$ if and only if $x^2 = (1-s)^2/(4s)$. That is possible when $(1-s)^2/(4s) \leq 1$ because $x = \sin\varphi$. Therefore, using the elementary considerations, we can find that if $s \in (3 - 2\sqrt{2}, 1)$, then the curve $F_s(e^{i\varphi})$, $\varphi \in [0, 2\pi)$ is concave and (3.10) is attained twice, when $\sin\varphi = (1-s)/(2\sqrt{s})$. In this case,

$$\max\{\Im\{F_s(e^{i\varphi})\} : \varphi \in [0, 2\pi)\} = \frac{1+s}{4\sqrt{s}(1-s)},$$

see Figure 3. Moreover, if $s \in [0, 3 - 2\sqrt{2}]$, then (3.10) is attained at one point only such that

$$\max\{\Im\{F_s(e^{i\varphi})\} : \varphi \in [0, 2\pi)\} = \Im\{F_s(e^{i\pi/2})\} = \frac{1}{1+s},$$

see Figures 2 and 3. □

Corollary 3.3 *If $0 \leq s \leq 3 - 2\sqrt{2} \approx 0.1715$, then $F_s \in \mathcal{K}$.*

By Lemma 2.1 the functions F_s , $s \in [0, 1]$, are starlike univalent, hence by the geometric interpretation (1.2) of a subordination under univalent functions, we obtain from Theorem 2.3 the following corollary.

Corollary 3.4 *If $0 \leq s \leq 6 - \sqrt{33}$ and $f \in \mathcal{H}$, then*

$$f(\Delta) \subset D(s) \implies L[f](\Delta) \subset L[F_s](\Delta). \quad (3.11)$$

Theorem 3.5 *Assume that $0 < s_i \leq 3 - 2\sqrt{2} \approx 0.1715$, $i = 1, 2$, and that the functions g_1 and g_2 are in the class \mathcal{H} with $g_1(0) = g_2(0) = 0$. If*

$$g_1(\Delta) \subset D(s_1) \quad \text{and} \quad g_2(\Delta) \subset D(s_2), \quad (3.12)$$

then

$$(g_1 * g_2)(\Delta) \subset D(s_1 s_2). \quad (3.13)$$

Proof Using Lemma 3.1, we may rewrite (3.12) as

$$g_1(\Delta) \subset F_{s_1}(\Delta) \quad \text{and} \quad g_2(\Delta) \subset F_{s_2}(\Delta). \quad (3.14)$$

By Corollary 3.3, F_s is convex univalent in Δ for $s \in [0, 3 - 2\sqrt{2}]$, so F_{s_1}, F_{s_2} are convex univalent in Δ . From (1.2) and from (3.14), we obtain

$$g_1 \prec F_{s_1} \quad \text{and} \quad g_2 \prec F_{s_2}. \quad (3.15)$$

By Theorem D, we get from (3.15)

$$g_1 * g_2 \prec F_{s_1} * F_{s_2}. \quad (3.16)$$

But

$$F_{s_1}(z) * F_{s_2}(z) = \frac{z}{1 - s_1 z^2} * \frac{z}{1 - s_2 z^2} = \frac{z}{1 - s_1 s_2 z^2} = F_{s_1 s_2}(z)$$

and hence by Lemma 3.1 the subordination (3.16) is equivalent to (3.13). \square

Theorem 3.6 Assume that $0 < s \leq 3 - 2\sqrt{2} \approx 0.1715$ and that the functions g_1 and g_2 are in the class \mathcal{H} with $g_1(0) = g_2(0) = 0$. Let $\mathfrak{R}(c, R)$ be the disc with center $c \in \mathbb{R}$ and radius R with $|c| < R$. If

$$g_1(\Delta) \subset D(s) \quad \text{and} \quad g_2(\Delta) \subset \mathfrak{R}(c, R), \quad (3.17)$$

then

$$(g_1 * g_2)(\Delta) \subset \frac{R^2 - c^2}{R} D\left(\frac{cs^2}{R^2}\right). \quad (3.18)$$

Proof Using Lemma 3.1, we may rewrite (3.17) as

$$g_1(\Delta) \subset F_{s_1}(\Delta) \quad \text{and} \quad g_2(\Delta) \subset H_{c,R}(\Delta), \quad (3.19)$$

where

$$H_{c,R}(z) = \frac{(R^2 - c^2)z}{R - cz} \quad (z \in \Delta)$$

maps Δ onto $\mathfrak{R}(c, R)$. By Corollary 3.3, F_s is convex univalent in Δ , also $H_{c,R}$ is convex univalent in Δ . Therefore, by Theorem D we obtain

$$\begin{aligned} (g_1 * g_2)(z) &\prec \frac{z}{1 - sz^2} * \frac{(R^2 - c^2)z}{R - cz} \\ &= \frac{R^2 - c^2}{R} \frac{z}{1 - sc^2 z^2 / R^2} \\ &= \frac{R^2 - c^2}{R} F_{sc^2/R^2}(z). \end{aligned}$$

The function F_{sc^2/R^2} is univalent as the convolution of a convex univalent function, so by (1.2) we obtain (3.18). \square

Theorem 3.7 Assume that $0 < s \leq 3 - 2\sqrt{2} \approx 0.1715$ and that the functions g_1 and g_2 are in the class \mathcal{H} with $g_1(0) = g_2(0) = 0$. Let $\mathfrak{L}(\varphi, d) = \{z \in \mathbb{C} : \Re\{ze^{-i\varphi}\} > -d\}$, $\varphi \in \mathbb{R}$, $d > 0$, be the half-plane. If

$$g_1(\Delta) \subset D(s) \quad \text{and} \quad g_2(\Delta) \subset \mathfrak{L}(\varphi, d), \quad (3.20)$$

then

$$(g_1 * g_2)(\Delta) \subset 2de^{-i\varphi} D(s). \quad (3.21)$$

Proof Using Lemma 3.1, we may rewrite (3.20) as

$$g_1(\Delta) \subset F_{s_1}(\Delta) \quad \text{and} \quad g_2(\Delta) \subset G_{\varphi, d}(\Delta), \quad (3.22)$$

where

$$G_{\varphi, d}(z) = \frac{2de^{-i\varphi} z}{1 - z} \quad (z \in \Delta)$$

maps Δ onto $\mathfrak{L}(\varphi, d)$. By Corollary 3.3, F_s is convex univalent in Δ , also $G_{\varphi, d}$ is convex univalent in Δ . Therefore, by Theorem D we obtain

$$\begin{aligned} (g_1 * g_2)(z) &< \frac{z}{1 - sz^2} * \frac{2de^{-i\varphi} z}{1 - z} \\ &= \frac{2de^{-i\varphi} z}{1 - sz^2} = 2de^{-i\varphi} F_s(z). \end{aligned}$$

Hence by (1.2) we obtain (3.21). □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

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